Proofs to three corollaries.

Corollary 3.1. In every triangle the following inequality holds

(3.1) $\frac{R}{r} \ge \frac{\sqrt{3}}{3}\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)$

with equality holding if and only if the triangle is equilateral.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let s be semiperimeter of $\triangle ABC$. We have

$\frac{R}{r} \ge \frac{\sqrt{3}}{3}\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right) \Leftrightarrow \frac{\sqrt{3}}{2r} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Leftrightarrow$

$\frac{\sqrt{3}}{2r} \ge \frac{ab+bc+ca}{abc} \Leftrightarrow \frac{\sqrt{3}}{2r} \ge \frac{ab+bc+ca}{4Rrs} \Leftrightarrow R\sqrt{3} \ge \frac{ab+bc+ca}{a+b+c}$.

Since $3(ab+bc+ca) \le (a+b+c)^2$ and $a+b+c \le 3\sqrt{3}R$ then

$\frac{ab+bc+ca}{a+b+c} \le \frac{(a+b+c)^2}{3(a+b+c)} = \frac{a+b+c}{3} \le \frac{3\sqrt{3}R}{3} = R\sqrt{3}$.

Equality in $3(ab+bc+ca) \le (a+b+c)^2$ and $a+b+c \le 3\sqrt{3}R$ occurs iff $a = b = c$.

Remark.

$\frac{\sqrt{3}}{2r} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Leftrightarrow \frac{2\sqrt{3}F}{2r} \ge \frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c} \Leftrightarrow$

$\frac{2\sqrt{3}rs}{2r} \ge h_a + h_b + h_c \Leftrightarrow h_a + h_b + h_c \le s\sqrt{3}$.★

Corollary 3.4. In every triangle we have

(**3.4**) $\frac{R}{r} \ge \frac{2}{9}(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

with equality holding if and only if the triangle is equilateral.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Since by (3.1) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{\sqrt{3}}{2r}$ and $a+b+c \le 3\sqrt{3}R$ then

$\frac{2}{9}(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \le \frac{2}{9}(a+b+c) \cdot \frac{\sqrt{3}}{2r} \le$

$\frac{2}{9} \cdot 3\sqrt{3}R \cdot \frac{\sqrt{3}}{2r} = \frac{R}{r}$.

Obviously that equlity conditions is the same as in (3.1).

Corollary 3.5. In every triangle we have

(**3.5**) $\frac{R}{r} \ge \frac{1}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right)$

with equality holding if and only if the triangle is equilateral.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let s be semiperimeter of $\triangle ABC$. Since $ab+bc+ca = s^2+4Rr+r^2$, $abc = 4Rrs$ then $\frac{R}{r} \ge \frac{1}{3}\sum\frac{b+c}{a} \Leftrightarrow \frac{R}{r} + 1 \ge \frac{1}{3}\sum\left(\frac{b+c}{a} + 1\right) \Leftrightarrow$

$\frac{R+r}{r} \ge \frac{(a+b+c)(ab+bc+ca)}{3abc} \Leftrightarrow \frac{R+r}{r} \ge \frac{2s(s^2+4Rr+r^2)}{3 \cdot 4Rrs} \Leftrightarrow$

$6R(R+r) \ge s^2+4Rr+r^2 \Leftrightarrow s^2 \le 6R^2+2Rr-r^2$.

Noting that $s^2 \le 4R^2+4Rr+3r^2$ (Gerretsen's Inequality) and $2r \le R$ (Euler's Inequality) we obtain $(6R^2+2Rr-r^2) - s^2 \ge$

$(6R^2 + 2Rr - r^2) - (4R^2 + 4Rr + 3r^2) = 2(R - 2r)(R + r) \geq 0.$
As cosequence of equality conditions in Gerretsen's and Euler's inequalities we obtain that equality holds iff the triangle is equilateral.